CE 228N: Introduction to the Theory of Plasticity: Homework 0

Instructor: Dr. Narayan Sundaram*

August 8, 2025

This is a refresher homework or a sort of 'pre-homework' on tensors, solid mechanics, and elasticity.

1. Show that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is linear in the displacement. You can start with either the tensorial definition $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u} \otimes \overleftarrow{\boldsymbol{\nabla}} + \overrightarrow{\boldsymbol{\nabla}} \otimes \mathbf{u}\right)$ or use a basis representation.

2. The compatibility condition for the infinitesimal strain tensor can be written in a compact form as $\overrightarrow{\boldsymbol{\nabla}} \times \boldsymbol{\varepsilon} \times \overleftarrow{\boldsymbol{\nabla}} = \mathbf{0}$. Note that due to the symmetry of $\boldsymbol{\varepsilon}$, one does not parentheses to indicate with $\times$ must be taken first.

   Using components in a Cartesian basis and basic definitions e.g.

$$\nabla \equiv \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \qquad \mathbf{a} \times \mathbf{b} \equiv \boldsymbol{\epsilon}_{ijk} a_j b_k \hat{\mathbf{e}}_i$$

   show that the compact form encodes all 6 of the St.-Venant compatibility equations. Here $\boldsymbol{\epsilon}_{ijk}$ is the Levi-Civita permutation symbol.

3. Invert the isotropic elastic constitutive relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$
$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

*Department of Civil Engineering, Indian Institute of Science

4. Consider the fourth-order symmetrizing identity tensor $\mathbb{I}^s$, whose action on any second-order tensor $\mathbf{A}$ is given by

$$\mathbb{I}^s : \mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$$

Show that the components of $\mathbb{I}^s$ in a basis are

$$I^s_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}\right)$$

5. As shown in class, the fourth-order isotropic elastic stiffness tensor is given by

$$\mathbb{C}_{iso} = 2\mu\,\mathbb{I}^s + \lambda\,\mathbb{1} \otimes \mathbb{1}$$

where $\mathbb{1}$ is the second-order identity tensor. Show that this tensor is indeed isotropic, i.e. its components are invariant under any orthogonal change of basis.

6. Write down an expression for the fourth-order isotropic elastic compliance tensor $\mathbb{S}_{iso}$. Show that $\mathbb{S}_{iso}$ and $\mathbb{C}_{iso}$ are indeed 'inverses' of each other in the sense that $\mathbb{S}_{iso} : (\mathbb{C}_{iso} : \boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}$ and $\mathbb{C}_{iso} : (\mathbb{S}_{iso} : \boldsymbol{\sigma}) = \boldsymbol{\sigma}$.